Quasilinear problems with the competition between convex and concave nonlinearities and variable potentials

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Abstract

The purpose of this paper is to prove some existence and non-existence theorems for the nonlinear elliptic problems of the form $-\Delta_p u = \lambda k(x) u^q \pm h(x) u^\sigma$ if $x \in \Omega$, subject to the Dirichlet conditions $u_1 = u_2 = 0$ on $\partial\Omega$. In the proofs of our results we use the sub-super solutions method and variational arguments. Related results as obtained here have been established in [Z. Guo and Z. Zhang, *$W^{1,p}$ versus C^1 local minimizers and multiplicity results for quasilinear elliptic equations*, Journal of Mathematical Analysis and Applications, Volume 286, Issue 1, Pages 32-50, 1 October 2003.] for the case $k(x) = h(x) = 1$. Our results reveal some interesting behavior of the solutions due to the interaction between convex-concave nonlinearities and variable potentials.

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1 Introduction and the main results

In this article we study the existence and non-existence of solutions for the quasilinear elliptic problems $(P_\lambda)_\pm$ of the following type

$$-\Delta_p u = \lambda k(x) u^q \pm h(x) u^\sigma \text{ if } x \in \Omega,\ u > 0 \text{ in } \Omega,\ u = 0 \text{ of } \partial\Omega \qquad ((P_\lambda)_\pm)$$

where λ is a positive real parameter, $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain with smooth boundary, $0 < q < p-1 < \sigma$, the variable weight functions k, $h \in L^\infty(\Omega)$ satisfy $ess\inf_{x\in\Omega} k(x) > 0$ and $ess\inf_{x\in\Omega} h(x) > 0$, and $\Delta_p u := \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$, $1 < p < \infty$ stands for the p-Laplacian operator.

We call a function $u : \Omega \to \mathbb{R}$ a solution of problems $(P_\lambda)_\pm$ if it belongs to the Sobolev space $W_0^{1,p}(\Omega)$ and such that

i) $u \geq 0$ a.e. on Ω and $u > 0$ on a subset of Ω with positive measure;

ii) for all $\varphi \in W_0^{1,p}(\Omega)$ the following identity holds

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi dx = \int_\Omega (\lambda k(x) u^q \pm h(x) u^\sigma)\varphi dx.$$

This kind of problems with convex and concave nonlinearities have been extensively studied and plays a central role in modern mathematical sciences, in the theory of heat conduction in electrically conduction materials, in the study of non-Newtonian fluids (see: Allegretto-Huang [1], Ambrosetti-Brezis-Cerami [2], Brezis-Nirenberg [3], Guo-Zhang [9], Figueiredo-Gossez-Ubilla [8] with their references). The basic work in our direction is the article [9] where Guo and Zhang have been considered the Dirichled problem

$$-\Delta_p u = \lambda u^q + u^\sigma \text{ if } x \in \Omega,\ u > 0 \text{ if } x \in \Omega,\ \ u = 0 \text{ if } x \in \partial\Omega,$$

where λ is a positive parameter, $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded domain with smooth boundary, $0 < q < p-1 < \sigma < p^* - 1$ inequality in which p^* represents for the Sobolev conjugate exponent of p, namely $p^* := Np/(N-p)$ if $1 < p < N$ and $p^* := \infty$ for $p \geq N$. We mention that in the work [9] the authors have been extended the results of Brezis and Nirenberg [3] obtained in the case $p = 2$.

Our main goal is to extend the results obtained in [9] to the more general problems $(P_\lambda)_\pm$.

The p-laplacian operator arises naturally in various contexts of physics, for instance, in non-Newtonian fluid theory, the quantity p is a characteristic of the medium. The case $1 < p < 2$ corresponds to pseudoplastics fluids and $p > 2$ arises in the consideration of dilatant fluids.

The main results are as follows:

Theorem 1.1. *Let $p > 1$. For all $0 < q < p-1 < \sigma < p^* - 1$ there exists a positive number λ^* such that for $\lambda \in (0, \lambda^*)$ the problem $(P_\lambda)_+$ has a minimal solution $u(\lambda)$ which is increasing with respect to λ. If $\lambda = \lambda^*$ the problem $(P_\lambda)_+$ has a solution. Moreover, problem $(P_\lambda)_+$ does not have any solution if $\lambda > \lambda^*$.*

Theorem 1.2. *Suppose $0 < q < p-1 < \sigma < p^* - 1$. Then there exists a positive number λ^* such that the problem $(P_\lambda)_-$ has at least one solution for $\lambda > \lambda^*$. Moreover, the problem $(P_\lambda)_-$ does not have any solution for $\lambda < \lambda^*$.*

Before we prove the main theorems, we need some additional results.

2 Preliminary results

The next result describes a regularity near the boundary for weak solutions to $((P_\lambda)_\pm$) and is developed by Lieberman in more general form than one presented here. For the interior regularity we advise the work of Tolksdorf [17] and DiBenedetto [7].

Lemma 2.1. *(in [12]) Let $\beta, \Lambda,\ M_0$ be positive constants with $\beta \leq 1$ and let Ω be a bounded domain in $\mathbb{R}^N$ with $C^{1,\beta}$ boundary. Suppose $b(x,r)$ satisfies the condition $|b(x,r)| \leq \Lambda$ for all (x,r) in $\partial\Omega \times [-M_0, M_0]$. If u is a bounded weak solution of the problem*

$$\Delta_p u + b(x,u) = 0 \text{ for } x \in \Omega,\ u = 0 \text{ on } \partial\Omega$$

with $|u| \leq M_0$ in Ω, then there is a positive constant $\alpha := \alpha(\alpha, \Lambda, N)$ such that u is in $C^{1,\alpha}\left(\overline{\Omega}\right)$. Moreover $|u|_{1+\alpha} \leq C(\alpha, \Lambda, M_0, N, \Omega)$.

We use in the proof the strong maximum principle of Vazquez.

Lemma 2.2. *(see [18]) Let Ω be a domain in $\mathbb{R}^N$ $(N \geq 1)$ and $u \in C^1(\Omega)$ such that $\Delta_p u \in L^2_{loc}(\Omega)$, $u \geq 0$ a.e. in Ω, $u \neq 0$, $\Delta_p u \leq \beta(u)$ a.e. in Ω with $\beta : [0, \infty) \to \mathbb{R}$ continuous, non-decreasing, $\beta(0) = 0$ and either $\beta(s) = 0$ for some $s > 0$ or $\beta(s) > 0$ for all $s > 0$ but*

$$\int_0^1 (j(S))^{-1/p} dS = \infty \text{ where } j(S) = \int_0^S \beta(t)dt.$$

Then if u does not vanish identically on Ω it is positive everywhere in Ω.

The following lemma has been obtained in Sakaguchi.

Lemma 2.3. *(see [15]) Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain with smooth boundary and let $u \in C^1(\overline{\Omega}) \cap W_0^{1,p}(\Omega)$ satisfy:*

$$\begin{aligned} -\Delta_p u &\geq 0 \text{ in } \Omega \text{ (in the weak sense)}, \\ u &> 0 \text{ in } \Omega \text{ and } u = 0 \text{ on } \partial\Omega. \end{aligned}$$

Then $\partial u/\partial n < 0$ on $\partial\Omega$ where n denotes the unit exterior normal vector to $\partial\Omega$.

The following comparison principle is proved in [15] (or consult some ideas of the proof in [16, Lemma 3.1.]).

Lemma 2.4. *Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain with smooth boundary and let $u, v \in W^{1,p}(\Omega)$ satisfy $-\Delta_p u \leq -\Delta_p v$ for $x \in \Omega$, in the weak sense. If $u \leq v$ on $\partial\Omega$ then $u \leq v$ in Ω.*

We prove Theorem 1.1 also by the method of sub- and super-solutions. To describe this method we introduce the problem

$$-\Delta_p u = \lambda k(x) u^q + h(x) u^\sigma \text{ for } x \in \Omega,\ u = 0 \text{ on } \partial\Omega, \tag{2.1}$$

where Ω, λ, k, q, h and σ are as above. We define $\underline{u} \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ to be a sub-solution of (2.1) if

$$\begin{aligned} -\Delta_p \underline{u} &\leq \lambda k(x) \underline{u}^q + h(x) \underline{u}^\sigma \ x \in \Omega, \text{ (in the weak sense)} \\ \underline{u} &= 0, \end{aligned}$$

and $\overline{u} \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ to be a super-solution of (2.1) if

$$\begin{aligned} -\Delta_p \overline{u} &\geq \lambda k(x) \overline{u}^q + h(x) \overline{u}^\sigma \ x \in \Omega, \text{ (in the weak sense)} \\ \overline{u} &= 0. \end{aligned}$$

Then the following result holds:

Lemma 2.5. *(see [5]) Suppose there exist a sub-solution $\underline{u}$ and a super-solution $\overline{u}$ of (2.1) in the above sense and that $\underline{u} \leq \overline{u}$. Then there exists a bounded weak solution u of the problem (2.1) such that $\underline{u} \leq u \leq \overline{u}$.*

We finally recall the following Picone's result for the p-Laplacian developed by Allegretto and Huang.

Lemma 2.6. *(see [1]) Let $v > 0$, $u \geq 0$ be differentiable. Denote*

$$R(u, v) = |\nabla u|^p - \nabla\left(\frac{u^p}{v^{p-1}}\right)|\nabla v|^{p-2}\nabla v.$$

Then $R(u, v) \geq 0$ and $R(u, v) = 0$ a.e. Ω if and only if $\nabla(u/v) = 0$ a.e. Ω, i.e. $u = kv$ for some constant k in each component of Ω, where Ω is bounded or unbounded, or the whole space $\mathbb{R}^N$.

3 Proof of the Theorem 1.1

Firstly, we prove that there exists $\lambda_0 > 0$ such that for all $\lambda \in (0, \lambda_0]$ the problem $(P_\lambda)_+$ has a solution. The argument relies on constructing a sub- and a super-solution with the properties from Lemma 2.5. In order to find a sub-solution, consider the problem

$$-\Delta_p u = \lambda k(x) u^q \text{ if } x \in \Omega,\ u > 0 \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega. \tag{3.1}$$

Then, by [6], problem (3.1) has a unique positive solution $w \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ with $\partial w/\partial n < 0$ on $\partial\Omega$. It is not difficult to prove that the function $\underline{u} := \varepsilon^{1/(p-1)} w$ is a sub-solution of problem $(P_\lambda)_+$ provided that $\varepsilon > 0$ is small enough. For this, it suffices to observe that

$$\varepsilon\lambda k(x) w^q \leq \lambda k(x) \varepsilon^{q/(p-1)} w^q + h(x) \varepsilon^{\sigma/(p-1)} w^\sigma \text{ in } \Omega$$

which is true for all $\varepsilon \in (0, 1)$. Let $v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ be the positive solution of

$$\begin{cases} -\Delta_p v = 1 \text{ in } \Omega \\ v = 0 \text{ on } \partial\Omega. \end{cases}$$

which exists and is unique from [10, Lemma 2.1.]. We prove that if $\lambda > 0$ is small enough then there is $M > 0$ such that $\overline{u} = M^{1/(p-1)} v$ is a super-solution of $(P_\lambda)_+$. Therefore it suffices to show that

$$M \geq \lambda k(x) \left[M^{1/(p-1)} v\right]^q + h(x) \left[M^{1/(p-1)} v\right]^\sigma. \tag{3.2}$$

In the next, we use some notations

$$A = \|k\|_{L^\infty} \cdot \|v\|_{L^\infty}^q \text{ and } B = \|h\|_{L^\infty} \cdot \|v\|_{L^\infty}^\sigma.$$

Thus by (3.2), it is enough to show that there is $M > 0$ such that

$$M \geq \lambda A M^{q/(p-1)} + B M^{\sigma/(p-1)}$$

that is equivalent to

$$1 \geq \lambda A M^{(q-p+1)/(p-1)} + B M^{(\sigma-p+1)/(p-1)}. \tag{3.3}$$

Consider the following mapping $(0, \infty) \ni t \to \lambda A t^{(q-p+1)/(p-1)} + B t^{(\sigma-p+1)/(p-1)}$. We also note that this function reaches its minimum value in $t = C\lambda^{(p-1)/(\sigma-q)}$, where

$$C = \left[AB^{-1}(p-1-q)(\sigma-p+1)^{-1}\right]^{(p-1)/(\sigma-q)}.$$

Moreover, the global minimum of this mapping is

$$\left[\left(AC^{(q-p+1)/(p-1)}+BC^{(\sigma-p+1)/(p-1)}\right)\right]\lambda^{(\sigma-p+1)/(\sigma-p)}.$$

This show that condition (3.3) is fulfilled for all $\lambda \in (0, \lambda_0]$ and $M = C\lambda^{(p-1)/(\sigma-q)}$, where λ_0 satisfies

$$\left[\left(AC^{(q-p+1)/(p-1)}+BC^{(\sigma-p+1)/(p-1)}\right)\right]\lambda_0^{(\sigma-p+1)/(\sigma-p)}=1.$$

Taking $\varepsilon > 0$ possibly smaller, we also note that the comparison principle announced in Lemma 2.4 implies $\varepsilon^{1/(p-1)}w \leq M^{1/(p-1)}v$. Thus, by Lemma 2.5 the problem $(P_\lambda)_+$ has at least one solution u_λ. Therefore, this solution is a critical point of the functional

$$u \longrightarrow \frac{1}{p}\int_\Omega |\nabla u|^p\,dx - \frac{\lambda}{q+1}\int_\Omega k(x)\,|u|^{q+1}\,dx - \frac{1}{\sigma+1}\int_\Omega h(x)\,|u|^{\sigma+1}\,dx$$

in the closed convex set $\left\{ u \in W_0^{1,p} \middle| \varepsilon^{1/(p-1)}w \leq u \leq M^{1/(p-1)}v \right\}$.

By choosing

$$\lambda^* = \sup\{\lambda > 0 |\ \text{problem } (P_\lambda)_+ \text{ has a solution}\},$$

we have from the definition of λ^* that problem $(P_\lambda)_+$ does not have any solution if $\lambda > \lambda^*$. In what follows we claim that λ^* is finite. Denote

$$m := \min\left\{ ess\inf_{x\in\Omega} k(x),\ ess\inf_{x\in\Omega} h(x) \right\}.$$

Clearly, $m > 0$. Let $\lambda' > 0$ be such that

$$m\left(\lambda' t^{q-p+1} + t^{\sigma-p+1}\right) > \lambda_1 \text{ for all } t \geq 0 \tag{3.4}$$

where λ_1 stands for the first eigenvalue of $(-\Delta_p)$ in $W_0^{1,p}(\Omega)$. Denote by φ_1 an eigenfunction of the p-Laplacian operator corresponding to λ_1. Then $\varphi_1 \in C^{1,\alpha}\left(\overline{\Omega}\right)$ and $\varphi_1 > 0$ in Ω as a consequence of the strong maximum principle of Vazquez (Lemma 2.2). We apply Picone's result, Lemma 2.6, to the function φ_1 and u_λ. We drop the parameter λ in the function u_λ and denote $u := u_\lambda$. Observe that $\frac{\varphi_1^p}{u^{p-1}}$ belongs to $W_0^{1,p}(\Omega)$ since u is positive in Ω and has nonzero outward derivative on the boundary because of the Hopf Lemma 2.3. Then for all $\lambda > \lambda'$ we have

$$\begin{aligned}
0 &\leq \int_\Omega |\nabla\varphi_1|^p\,dx - \int_\Omega \nabla\left(\frac{\varphi_1^p}{u^{p-1}}\right)|\nabla u|^{p-2}\,\nabla u dx \\
&= \int_\Omega |\nabla\varphi_1|^p\,dx - \int_\Omega \frac{\varphi_1^p}{u^{p-1}}\Delta_p u dx \\
&= \int_\Omega |\nabla\varphi_1|^p\,dx - \int_\Omega \frac{\varphi_1^p}{u^{p-1}}\left(\lambda k(x)\,u^q + h(x)\,u^\sigma\right)dx \\
&< \int_\Omega \lambda_1\varphi_1^p dx - \int_\Omega m\left[\lambda k(x)\,u^{q-p+1} + h(x)\,u^{\sigma-p+1}\right]\varphi_1^p dx \\
&< \int_\Omega \lambda_1\varphi_1^p dx - \int_\Omega m\left[\lambda' u^{q-p+1} + u^{\sigma-p+1}\right]\varphi_1^p dx \\
&= \int_\Omega \left[\lambda_1 - m\left(\lambda' u^{q-p+1} + u^{\sigma-p+1}\right)\right]\varphi_1^p dx < 0.
\end{aligned}$$

Thus we get a desired contradiction. As a conclusion we obtain the following result $\lambda^* \leq \lambda' < \infty$ which proves our claim. Let as now prove that u_λ is a minimal solution of the problem $(P_\lambda)_+$.

By the definition of λ^* there exists $\overline{\lambda} < \lambda$ such that $\overline{\lambda} < \lambda^*$ and $(P_{\overline{\lambda}})_+$ has a positive solution $u_{\overline{\lambda}}$. The rest of the argument is based on the standard monotone iteration. Consider the sequence $(u_n)_{n\geq 0}$ defined by $u_0 = w$ (where w is the unique solution of (3.1)) and u_n the solution of the problem

$$\begin{aligned} -\Delta_p u_n &= \lambda k(x)\, u_{n-1}^q + h(x)\, u_{n-1}^\sigma, \text{ if } x \in \Omega \\ u_n(x) &> 0, \text{ if } x \in \Omega \\ u_n(x) &= 0, \text{ if } x \in \partial\Omega \end{aligned}$$

which exists and is unique from the results in [11] (see also arguments in [9]). By using the comparison principle, it is not hard to show that

$$u_0 = w \leq u_1 \leq ... \leq u_n \leq u_{n+1} \leq u_{\overline{\lambda}} \text{ in } \Omega. \tag{3.5}$$

In fact, it follows again by the above cited comparison principle applied to the problem

$$\begin{aligned} -\Delta_p u_0 &= \lambda k(x)\, u_0^q \leq \lambda k(x)\, u_0^q + h(x)\, u_0^\sigma = -\Delta_p u_1 \text{ in } \Omega, \\ u_0 &= u_1 = 0 \text{ on } \partial\Omega \end{aligned}$$

that $u_0 \leq u_1$ in Ω. Similarly, one can show by using the same Lemma 2.4 that $u_1 \leq u_2$ in Ω. In particular, for all $x \in \Omega$ the sequence $(u_n)_{n\geq 0}$ is a nondecreasing sequence which is bounded and therefore $u_n \leq U$ for any positive solution U of $(P_\lambda)_+$. Using the relation (3.5), the decay property of $u_{\overline{\lambda}}$ and a standard diagonalization procedure we get a subsequence converging to a solution u_λ of $(P_\lambda)_+$, satisfying $u_\lambda \leq u_{\overline{\lambda}}$ and $u_\lambda \leq U$ for any arbitrary solution U of problem $(P_\lambda)_+$. The conclusion then follow. At this stage it is easy to deduce that the mapping u_λ is increasing with respect to λ. We consider u_{λ_1}, u_{λ_2} with $0 < \lambda_1 < \lambda_2 < \lambda^*$. Since

$$-\Delta_p u_{\lambda_2} = \lambda_2 k(x)\, u_{\lambda_2}^q + h(x)\, u_{\lambda_2}^\sigma > \lambda_1 k(x)\, u_{\lambda_2}^q + h(x)\, u_{\lambda_2}^\sigma$$

then u_{λ_2} is a super-solution of problem $(P_{\lambda_1})_+$. The argument used above may be used to construct a sequence $(u_n)_{n\geq 0}$ such that $0 < u_{n-1} < u_n < u_{\lambda_2}$ converging to a solution U of $(P_{\lambda_1})_+$ with $U < u_{\lambda_2}$ and therefore $u_{\lambda_1} \leq U < u_{\lambda_2}$ by the minimality of u_{λ_1}. This proves our claim.

It remain to show that problem $(P_\lambda)_+$ has a solution if $\lambda = \lambda^*$. For this purpose it is enough to prove that

$$u_\lambda \text{ is bounded in } W_0^{1,p}(\Omega) \text{ as } \lambda \to \lambda^*. \tag{3.6}$$

Thus, by passing to a suitable subsequence if necessary, we may assume

$$u_\lambda \to u^* \text{ in } W_0^{1,p}(\Omega) \text{ as } \lambda \to \lambda^*,$$

which implies that u^* is a weak solution of $(P_\lambda)_+$ provided that $\lambda = \lambda^*$. Moreover since the mapping $\lambda \to u_\lambda$ is increasing, it follows that $u^* \geq 0$ a.e. on Ω and $u^* > 0$ on a subset of Ω with positive measure. As we mentioned, it is often advantageous to work with u instead of u_λ. A key ingredient of the proof is that all solutions u have negative energy. More precisely, if $E : W_0^{1,p}(\Omega) \to \mathbb{R}$ is defined by

$$E(u) = \frac{1}{p}\int_\Omega |\nabla u|^p\, dx - \frac{\lambda}{q+1}\int_\Omega k(x)\, |u|^{q+1}\, dx - \frac{1}{\sigma+1}\int_\Omega h(x)\, |u|^{\sigma+1}\, dx$$

then

$$E(u) < 0 \text{ for all } \lambda \in (0, \lambda^*). \tag{3.7}$$

We do it in the following steps:

Step 1) the solution u satisfies

$$\int_\Omega \left\{ |\nabla u|^p - [\lambda q/(p-1)]\, k(x)\, u^{q+1} + [\sigma/(p-1)]\, h(x)\, u^{\sigma+1} \right\} dx \geq 0. \tag{3.8}$$

This follows by the same arguments from [9, Lemma 3.7.].

Step 2) Since u is a solution of $(P_\lambda)_+$ we have

$$\int_\Omega |\nabla u|^p\, dx = \int_\Omega \lambda k(x)\, u^{q+1} dx + \int_\Omega h(x)\, u^{\sigma+1} dx. \tag{3.9}$$

Plugging relation (3.8) into (3.9) we have

$$\lambda (p-1-q) \int_\Omega k(x)\, u^{q+1} dx \geq (\sigma+1-p) \int_\Omega h(x)\, u^{\sigma+1} dx \tag{3.10}$$

In particular, it follows from these two latest relations that

$$\begin{aligned} E(u) &= \lambda \left(\frac{1}{p} - \frac{1}{q+1} \right) \int_\Omega k(x)\, u^{q+1} dx + \left(\frac{1}{p} - \frac{1}{\sigma+1} \right) \int_\Omega h(x)\, u^{\sigma+1} dx \\ &= -\lambda \frac{p-1-q}{p(q+1)} \int_\Omega k(x)\, u^{q+1} dx + \frac{\sigma+1-p}{p(\sigma+1)} \int_\Omega h(x)\, u^{\sigma+1} dx \\ &\leq -\lambda \frac{p-1-q}{p(q+1)} \int_\Omega k(x)\, u^{q+1} dx + \lambda \frac{p-1-q}{p(\sigma+1)} \int_\Omega h(x)\, u^{\sigma+1} dx \leq 0. \end{aligned}$$

Thus, by combining (3.7) and (3.8), sobolev embedings, and using the fact that $k,\, h \in L^\infty(\Omega)$ it follows

$$\sup \left\{ \|u_\lambda\|_{W_0^{1,p}(\Omega)} \middle| \lambda < \lambda^* \right\} < \infty$$

and so (3.6) is finished. This complete the proof of Theorem 1.1.

4 Proof of the Theorem 1.2

The study of existence of solutions to problem $(P_\lambda)_-$ is done by looking for critical points of the functional $F_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$F_\lambda(u) = \frac{1}{p} \int_\Omega |\nabla u|^p\, dx - \frac{\lambda}{q+1} \int_\Omega k(x)\, |u|^{q+1}\, dx + \frac{1}{\sigma+1} \int_\Omega h(x)\, |u|^{\sigma+1}\, dx.$$

In the next we adopt the following notations

$$\|u\| := \left(\int_\Omega |\nabla u|^p\, dx \right)^{1/p},\ \|u\|_{q+1} := \left(\int_\Omega |u|^{q+1}\, dx \right)^{1/(q+1)},\ \|u\|_{\sigma+1} := \left(\int_\Omega |u|^{\sigma+1}\, dx \right)^{1/(\sigma+1)}.$$

We prove that F_λ is coercive. In order to verify this claim, we first observe that

$$F_\lambda(u) \geq \frac{1}{p} \|u\|^p - C_1 \|u\|_{q+1}^{q+1} + C_2 \|u\|_{\sigma+1}^{\sigma+1},$$

where

$$C_1 = \frac{\lambda}{q+1} \|k\|_{L^\infty} \text{ and } C_2 = \frac{1}{\sigma+1} ess \inf_{x\in\Omega} h(x)$$

are positive constants. Since $q < \sigma$, a simple calculation shows that the mapping

$$(0,\infty) \ni t \to C_1 t^{q+1} - C_2 t^{\sigma+1}$$

attains its global minimum $m < 0$ at

$$t = \left[\frac{C_2(q+1)}{C_1(\sigma+1)}\right]^{1/(\sigma-q)}.$$

So we conclude that

$$F_\lambda(u) \geq \frac{1}{p}\|u\|^p + m,$$

and hence $F_\lambda(u) \to \infty$ as $\|u\| \to \infty$ whish finished the proof that F_λ is coercive. Let $(u_n)_{n\geq 0}$ be a minimizing sequence of F_λ in $W_0^{1,p}(\Omega)$. The coercivity of F_λ implies the boundedness of u_n in $W_0^{1,p}(\Omega)$. Then, up to a subsequence if necessary, we may assume that there exists u in $W_0^{1,p}(\Omega)$ non-negative such that $u_n \overset{n\to\infty}{\rightharpoonup} u$ weakly in $W_0^{1,p}(\Omega)$. We remark that the function u can be non-negative due to $F_\lambda(u) = F_\lambda(|u|)$. Standard arguments based on the lower semi-continuity of the energy functional show that u is a global minimizer of F_λ and therefore is a solution in the sense of distributions of $(P_\lambda)_-$.

In what follows we claim that the weak limit u is a non-negative weak solution of problem $(P_\lambda)_-$ if $\lambda > 0$ is large enough. We first observe that $F_\lambda(0) = 0$. So, in order to prove that the non-negative solution is non-trivial, it suffices to prove that there exists $\Lambda > 0$ such that

$$\inf_{u\in W_0^{1,p}(\Omega)} F_\lambda(u) < 0 \text{ for all } \lambda > \Lambda.$$

For this purpose we consider the variational problem with constraints,

$$\Lambda = \inf\left\{\frac{1}{p}\int_\Omega |\nabla v|^p\,dx + \frac{1}{\sigma+1}\int_\Omega h(x)|v|^{\sigma+1}\,dx \,\middle|\, v \in W_0^{1,p}(\Omega) \text{ and } \frac{1}{q+1}\int_\Omega k(x)|v|^{q+1}\,dx = 1\right\}. \tag{4.1}$$

Let $(v_n)_{n\geq 0}$ be an arbitrary minimizing sequence for this problem. Then v_n is bounded, hence we can assume that it weakly converges to some $v \in W_0^{1,p}(\Omega)$ with

$$\frac{1}{q+1}\int_\Omega k(x)|v|^{q+1}\,dx = 1 \text{ and } \Lambda = \frac{1}{p}\int_\Omega |\nabla v|^p\,dx + \frac{1}{\sigma+1}\int_\Omega h(x)|v|^{\sigma+1}\,dx.$$

Thus

$$F_\lambda(v) = \Lambda - \lambda \text{ for all } \lambda > \Lambda.$$

Set

$$\lambda^* := \inf\{\lambda > 0|\ \text{problem } (P_\lambda)_- \text{ admits a nontrivial weak solution}\} \geq 0.$$

The above remarks show that $\Lambda \geq \lambda^*$ and that problem $(P_\lambda)_-$ has a solution for all $\lambda \geq \Lambda$. We now argue that problem $(P_\lambda)_-$ has a solution for all $\lambda > \lambda^*$. Fixed $\lambda > \lambda^*$, by the definition of λ^*, we can take $\mu \in (\lambda^*, \lambda)$ such that F_μ has a nontrivial critical point $u_\mu \in W_0^{1,p}(\Omega)$. Since $\mu < \lambda$,

it follows that u_μ is a sub-solution of problem $(P_\lambda)_-$. We now want to construct a super-solution that dominates u_μ. For this purpose we consider the constrained minimization problem

$$\inf\left\{F_\lambda(v),\ v\in W_0^{1,p}(\Omega)\ \text{ and } v\geq u_\mu\right\}. \tag{4.2}$$

From the previous arguments, used to treat (4.1) follows that problem (4.2) has a solution $u_\lambda > u_\mu$. Moreover, u_λ is a solution of problem $(P_\lambda)_-$ for all $\lambda>\lambda^*$. With the arguments developed in [9] we deduce that problem $(P_\lambda)_-$ has a solution if $\lambda=\lambda^*$. The same monotonicity arguments as above show that $(P_\lambda)_-$ does not have any solution if $\lambda<\lambda^*$. Fix $\lambda>\lambda^*$. It remains to argue that the non-negative weak solution u is, in fact, positive. Indeed, using Moser iteration, we obtain that $u\in L^\infty(\Omega)$. Once $u\in L^\infty(\Omega)$ it follows by Lemma 2.1 that u is a $C^{1,\alpha}\left(\overline{\Omega}\right)$ solution of problem $(P_\lambda)_-$ provided for some α. Invoking the nonlinear strong maximum principle of Vazquez (Lemma 2.2), since u is a non-negative smooth weak solution of the differential inequality

$$-\Delta_p u + h(x)\,u^\sigma \geq 0 \text{ in } \Omega,$$

we deduce that u is positive everywhere in Ω. The proof of Theorem 1.2 is completed.

The extension of the above results to all space $\mathbb{R}^N$ or to the nonlinearities depending on the gradient ∇u requires some further nontrivial modifications and will be considered in a future work. We anticipate that the methods and concepts here can be extended to systems or when in discussion are more general linear/non-linear operators as well.

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